ELSEVIER

Journal of Geometry and Physics 44 (2002) 41–51

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Harmonic twistor formalism and transgression on hyperkähler manifolds

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Abstract

In this paper we continue our study of the fourth-order transgression on hyperähler manifolds introduced in the previous paper. We give a local construction for the fourth-order transgression of the Chern character form of an arbitrary vector bundle supplied with a self-dual connection on a four-dimensional hyperkähler manifold. The construction is based on the harmonic twistor formalism. Remarkably, the resulted expression for the fourth-order transgression is given in terms of the determinant of the $\bar{\partial}$-operator defined on fibers of the twistor fibration.

MSC: 53C26; 58J52; 53Z05

Subj. Class: Differential geometry

Keywords: Hyperkähler; Yang–Mills; Twistors; Chiral determinant; Transgression

1. Introduction

In this paper we continue our study of the transgression of characteristic classes of hyperholomorphic bundles on hyperkähler manifolds [2]. In the previous paper the global construction for the fourth-order transgression of the Chern character form on a compact hyperkähler manifold was proposed. In addition, the explicit expression for the transgression of the Chern character arising in the application of the local families index theorem was found. This construction was local over the base of the fibration. It is natural to look for a local derivation of the transgression of the Chern character forms for an arbitrary hyperholomorphic bundle. In this paper we give the general local construction for an arbitrary hyperholomorphic bundle on a four-dimensional hyperkähler manifold. Note that

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PII: S0393-0440(02)00049-9

in $d = 4$ the condition on a hermitian bundle to be hyperholomorphic is equivalent to the antiself-duality condition on the corresponding connection. We propose an explicit expression for the fourth-order transgression $T(\mathcal{E})$ of the top degree part of the Chern character form for an arbitrary vector bundle $\mathcal{E}$ supplied with a self-dual connection. The construction is local and thus is applicable to an arbitrary four-dimensional hyperkähler manifold M. Locally the Chern character form is exact and we have:

$$ch_{[2]}(\mathcal{E}) = \mathrm{vol}_M \Delta^2 T$$

for a volume form vol_M. Remarkably, the explicit expression for $T(\mathcal{E})$ is non-trivial even for a linear bundle $\mathcal{E}$.

In our derivation we essentially use the harmonic twistor approach, the variant of the twistor formalism developed in [3,4,8]. In twistor approach [9] one codes the information about self-dual connections on a vector bundle in terms of holomorphic structures on a bundle over the twistor fibration $Z_M \to M$ with a fiber being S^2. Remarkably, the proposed expression for $T(\mathcal{E})$ is given in terms of the determinant of the $\bar{\partial}_A$-operator in the sense of Quillen [7] acting on sections of the holomorphic bundle restricted to the fibers. This implies that the results of this paper may be connected with the local families index for the twistor fibration.

Rather straightforwardly, the construction described in this paper may be generalized to hyperkähler manifolds of an arbitrary dimension. We are going to discuss the general construction connecting the approaches of this paper and [2] the future publication.

2. Harmonic twistor formalism

In this section we give a short account of the harmonic formalism closely following the presentation given in [3,4]. Let M be a four-dimensional manifold. Holonomy group in four dimensions is a tensor product $Sp(1) \otimes Sp(1)$ and T^*M naturally splits:

$$T^*M = \mathcal{H}_L \otimes \mathcal{H}_R,$$

where $\mathcal{H}_{L,R}$ are $Sp(1)$-bundles over M with the connection forms $\omega^{\dot\alpha}_{(L,R)\dot\beta}$ such that

$$\nabla h^{\dot\alpha} = h^{\dot\beta}\omega^{\dot\alpha}_{L\dot\beta}, \qquad \nabla e^{\alpha} = e^{\beta}\omega^{\alpha}_{R\beta}.$$

In this notations $\theta^{\dot\alpha\alpha} = h^{\dot\alpha} \otimes e^{\alpha}$ is the basis of 1-forms.

Let Z be a total space of $\mathcal{H}_L \setminus 0$. Let us introduce harmonic variables $u^{\pm\dot\alpha}$, $u^{\pm}_{\dot\alpha}$ with the following properties:

$$u^+_{\dot\alpha} = u^{+\dot\beta}\epsilon_{\dot\alpha\dot\beta}, \qquad u^{+\dot\alpha} = u^+_{\dot\beta}\epsilon^{\dot\alpha\dot\beta}, \tag{1}$$

$$u^-_{\dot\alpha} = u^{-\dot\beta}\epsilon_{\dot\beta\dot\alpha}, \qquad u^{-\dot\alpha} = u^-_{\dot\beta}\epsilon^{\dot\beta\dot\alpha}, \tag{2}$$

where $\epsilon^{\dot\alpha\dot\beta} = -\epsilon_{\dot\alpha\dot\beta} = \epsilon_{\dot\beta\dot\alpha}$.

They define a frame in the bundle $\mathcal{H}_L$. We will consider the spherical bundle defined by the condition:

$$u^+_{\dot\alpha} u^{+\dot\alpha} = 0, \qquad u^+_{\dot\alpha} u^{-\dot\alpha} = 1, \tag{3}$$

supplied with the reality condition:

$$\overline{u^{\pm\dot\alpha}} = u^{\pm}_{\dot\alpha}. \tag{4}$$

Let M be a hyperkähler manifold. Then one could chose the trivial connection on $\mathcal{H}_L$ ($\omega^{\dot\alpha}_{L\dot\beta} = 0$). Let us introduce the bases of vertical forms $\theta^{\pm\dot\alpha}$ and horizontal forms $\eta^{\pm\alpha}$ with respect to projection $Z \to M$ as follows:

$$\theta^{\pm\dot\alpha} = du^{\pm\dot\alpha}, \qquad \eta^{\pm\alpha} = u^{\pm\dot\alpha}\theta^{\dot\alpha\alpha}.$$

The variables $u^{\pm\dot\alpha}$ parameterize complex structures on $T^*_x M$ compatible with the hyperkähler structure. At each point $(x, u) \in Z$ the forms $\eta^{+\alpha}$ $(\eta^{-\alpha})$ span the distribution of the holomorphic (antiholomorphic) forms with respect to the complex structure u on $T^*_x M$.

We will be interested in the local properties of the self-dual connections on vector bundles. Thus we could consider the flat space $M = \mathbb{R}^4$ with the standard metric $g_{\mu\nu} = \delta_{\mu\nu}$ in coordinates x^μ. As the sections of $Sp(k) \times Sp(1)$-bundle, the coordinates x^μ may be written as $x^{\alpha\dot\alpha}$ with the reality condition $x^{\alpha\dot\alpha} = \overline{x_{\alpha\dot\alpha}}$, where

$$x_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}x^{\beta\dot\beta} = \epsilon_{\beta\alpha}\epsilon_{\dot\beta\dot\alpha}x^{\beta\dot\beta}, \tag{5}$$

$$x^{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}x_{\beta\dot\beta} = \epsilon_{\beta\alpha}\epsilon_{\dot\beta\dot\alpha}x_{\beta\dot\beta}. \tag{6}$$

For the complex structure defined by u the antiholomorphic coordinates are:

$$x^{\pm\alpha} = u^{\pm}_{\dot\alpha}x^{\alpha\dot\alpha}, \tag{7}$$

$$x^{\pm}_{\alpha} = u^{\pm\dot\alpha}x_{\alpha\dot\alpha}. \tag{8}$$

Now the reality condition can be written as follows: $\overline{x^{\pm}_{\alpha}} = x^{\mp\alpha}$. In fact

$$\overline{x^{\pm}_{\alpha}} = \overline{u^{\pm\dot\alpha}}\,\overline{x_{\alpha\dot\alpha}} = u^{\mp}_{\dot\alpha}x^{\alpha\dot\alpha} = x^{\mp\alpha}.$$

Using (1), (2), (5) and (6), one gets

$$x^{+\alpha} = x^{+}_{\beta}\epsilon^{\beta\alpha}, \qquad x^{+}_{\alpha} = x^{+\beta}\epsilon_{\beta\alpha}, \tag{9}$$

$$x^{-\alpha} = x^{-}_{\beta}\epsilon^{\alpha\beta}, \qquad x^{-}_{\alpha} = x^{-\beta}\epsilon^{\alpha\beta}. \tag{10}$$

For the differential operators $\partial^{\pm}_{\alpha} \equiv u^{\pm\dot\beta}\partial_{\alpha\dot\beta}$ we have the following simple relation:

$$\partial^{\pm}_{\alpha} = \frac{\partial}{\partial x^{\mp\alpha}}. \tag{11}$$

Taking into account the normalization conditions (3), we get the expression for the Laplace operator:

$$\Delta \equiv \partial^{\mu}\partial_{\mu} = 2\partial^{\pm}_{\alpha}\partial^{\mp\alpha}. \tag{12}$$

In the following we will use the realization of $sl(2)$ algebra:

$$[D^{++}; D^{--}] = D^0, \tag{13}$$

$$[D^0; D^{\pm\pm}] = \pm 2D^{\pm\pm}, \tag{14}$$

by the first order differential operators:

$$D^{++} = u^{+\dot\alpha}\frac{\partial}{\partial u^{-\dot\alpha}}, \tag{15}$$

$$D^{--} = u^{-\dot\alpha}\frac{\partial}{\partial u^{+\dot\alpha}}, \tag{16}$$

$$D^0 = u^{+\dot\alpha}\frac{\partial}{\partial u^{+\dot\alpha}} - u^{-\dot\alpha}\frac{\partial}{\partial u^{-\dot\alpha}}, \tag{17}$$

with the properties:

$$D^{++}u^{+\dot\alpha} = 0, \tag{18}$$

$$D^{++}u^{-\dot\alpha} = u^{+\dot\alpha}. \tag{19}$$

One could introduce the formal analog of integration of the functions of the harmonic variables as follows. Let $f^{(q)}(u)$ be a function of the charge q ($D^0(f^{(q)}) = qf^{(q)}$). Then it has the following expansion:

$$f^{(q)}(x; u^{\pm}) = \sum f^{(\dot\alpha_1,\dots,\dot\alpha_{n+q}\dot\beta_1,\dots,\dot\beta_n)}u^+_{\dot\alpha_1},\dots,u^+_{\dot\alpha_{n+q}}u^-_{\dot\beta_1},\dots,u^-_{\dot\beta_n}. \tag{20}$$

The integration may be defined by the conditions:

$$\int \mathrm{d}^2u1 = 1, \tag{21}$$

$$\int \mathrm{d}^2uu^+_{(i_1,\dots,i_n}u^-_{j_1,\dots,j_m)} = 0, \quad n+m>0. \tag{22}$$

Note that the integral of the function with non-zero-charge is zero. Integration rules defined by (21) have the usual property to be zero for a total derivative:

$$\int \mathrm{d}^2uD^{++}f = 0. \tag{23}$$

Thus defined integration is equivalent to the usual integration over the sphere S^2 in terms of coordinates $u^{\pm,\dot\alpha}$ but has a virtue to be defined algebraically.

Consider a hermitian vector bundle $\mathcal{E}$ on M with a connection ∇. For the holomorphic structure defined by u the holomorphic and antihiolomorphic parts of the connection are given by:

$$\nabla^{\pm}_{\alpha} = u^{\pm\dot\alpha}\nabla_{\alpha\dot\alpha}, \qquad A^{\pm}_{\alpha} = u^{\pm\dot\alpha}A_{\alpha\dot\alpha}. \tag{24}$$

The curvature of the connection ∇ has the representation:

$$F_{\alpha\dot\alpha\beta\dot\beta} = \epsilon_{\dot\alpha\dot\beta}f_{\alpha\beta} + \Omega_{\alpha\beta}f_{\dot\alpha\dot\beta},$$

where ϵ, Ω are antisymmetric, f is symmetric.

Let the connection ∇ on $\mathcal{E}$ be self-dual. Thus the self-dual part of the curvature $f_{\dot{\alpha}\dot{\beta}}$ is zero and we have:

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta}. \tag{25}$$

From the conditions (3) it follows that the connection is integrable on the holomorphic and antiholomorphic hyperplanes:

$$[\nabla^{\pm}_{\alpha}, \nabla^{\pm}_{\beta}] = 0, \tag{26}$$

$$[\nabla^{+}_{\alpha}, \nabla^{-}_{\beta}] = f_{\alpha\beta}. \tag{27}$$

The operators $\nabla^{\pm}_{\dot{\alpha}}$ which lead to self-dual connection may be characterized by the following set of equations:

$$[\nabla^{\pm}_{\alpha}; \nabla^{\pm}_{\beta}] = 0, \tag{28}$$

$$[D^{\pm\pm}; \nabla^{\pm}_{\alpha}] = 0, \tag{29}$$

$$[D^{0}; \nabla^{\pm}_{\alpha}] = \pm\nabla^{\pm}_{\alpha}. \tag{30}$$

The main advantage of the representation (28)–(30) is the possibility to use the u-dependent gauge transformations for finding explicit solutions of the self-duality conditions. This goes as follows. Locally the first equation (28) allows to represent the positive part of the harmonic connection as the pure gauge with zero-charge gauge parameter U:

$$\nabla^{+}_{\alpha} = U^{-1}\partial^{+}_{\alpha}U, \tag{31}$$

$$q(\nabla^{+}_{\alpha}) = 1 \Rightarrow q(U(x,u)) = 0. \tag{32}$$

After the gauge transformation with parameter $U(x,u)^{-1}$ we get the set of covariant derivatives:

$$\nabla^{+}_{\alpha} = \partial^{+}_{\alpha}, \tag{33}$$

$$\mathcal{D}^{++} = D^{++} + V^{++} = D^{++} + UD^{++}U^{-1}, \tag{34}$$

$$\mathcal{D}^{0} = D^{0}. \tag{35}$$

Now the only constraint on the function V^{++} with $q = 2$ comes from (29):

$$\frac{\partial}{\partial x^{-\alpha}} V^{++} = 0. \tag{36}$$

The solution of this equation obviously is given by an arbitrary function of $X^{+\alpha}$, $u^{\pm\dot{\alpha}}$ with the total charge $q = 2$. Taking into account the properties of the integral (21) we could reconstruct the gauge field from the solution of the Eq. (36):

$$A_{\alpha\dot{\alpha}} = \int \mathrm{d}^2u\, u^{-}_{\dot{\alpha}}(U^{-1}\partial^{+}_{\alpha}U). \tag{37}$$

As an example consider the following matrix-valued function corresponding to the gauge group $Sp(1)$ [5,6]:

$$(V^{++})_i^j = \frac{x^{+j}x_i^+}{\rho^2}. \tag{38}$$

This leads to:

$$(U)_i^j = \left(1 + \frac{x^2}{\rho^2}\right)^{-1/2} \left(\delta_i^j + \frac{x^{+j}x_i^-}{\rho^2}\right), \tag{39}$$

$$A_{\alpha\dot{\alpha}i}^j = \frac{1}{\rho^2 + x^2}\left(\frac{1}{2}x_{\alpha\dot{\alpha}}\delta_i^j + \epsilon_{i\alpha}x_{\dot{\alpha}}^j\right). \tag{40}$$

Thus we get the one-instanton solution [1] with the center at $x = 0$ and the size ρ.

3. Forth order transgression of the second Chern class

According to the general considerations in [2] it is natural to expect that locally the Chern character form of a hyperholomorphic bundle $\mathcal{E}$ over a hyperkähler manifold M admits the forth order transgression:

$$ch(\mathcal{E}) = dd_I d_J d_K(\tau(\mathcal{E})), \tag{41}$$

where $d_I = IdI^{-1}, d_J = JdJ^{-1}, d_K = KdK^{-1}$ are exterior derivative operators twisted by the compatible complex structures I, J, K. For four-dimensional hyperkähler manifold this relation simplifies:

$$ch_{[2]}(\mathcal{E}) = \mathrm{vol}_M \Delta^2 T(\mathcal{E}). \tag{42}$$

Here $ch_{[2]}$ is a degree four component of the Chern character, vol_M is the volume form on M and Δ is the Laplace operator.

In this paper we prove the relation (42) using the harmonic twistor formalism and give the representation for T in terms of the determinant of the first order differential operator:

Theorem 1. *Let $\mathcal{E}$ be a hermitian vector bundle on a hyperkähler four-dimensional manifold M with a self-dual connection ∇ and the curvature form $F = \nabla^2$. The following local expression for the top degree part of the Chern character form holds*:

$$ch_{[2]}(\mathcal{E}) = -\frac{1}{8\pi^2}\mathrm{Tr}(F \wedge F) = \mathrm{vol}_M \Delta^2 T(\mathcal{E}), \tag{43}$$

$$T(\mathcal{E}) = \frac{1}{16\pi^2}\log\left(\frac{\mathrm{Det}(D^{++} + V^{++})}{\mathrm{Det}(D^{++})}\right). \tag{44}$$

The determinant here is essentially the determinant of the operator $\bar{\partial}$ (in holomorphic parameterization we have $D^{++} \sim \partial_{\bar{z}}$).

In our parameterization the formulas (A.5) and (A.6) from the Appendix A take the form:

$$\delta \log \mathrm{Det}(D^{++} + V^{++}) = \frac{1}{2} \int \mathrm{d}^2 u(\delta V^{++} V^{--}(V^{++})), \tag{45}$$

with the condition:

$$D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0. \tag{46}$$

Locally one could represent A^{+}-component as a pure gauge:

$$A^{+}_{\alpha} = U^{-1}\partial^{+}_{\alpha}(U). \tag{47}$$

In this parameterization we have:

$$V^{++} = 0, \tag{48}$$

$$V^{--} = 0, \tag{49}$$

$$\nabla^{-}_{\alpha} = [D^{--}, \nabla^{+}_{\alpha}] = U^{-1}[\mathcal{D}^{--}, \partial^{+}_{\alpha}]U, \tag{50}$$

$$A^{-}_{\alpha} = -U^{-1}\partial^{+}_{\alpha}(V^{--})U + U^{-1}\partial^{-}_{\alpha}(U), \tag{51}$$

$$f_{\alpha\beta} = U^{-1}[\partial^{+}_{\alpha}, \partial^{-}_{\beta} - \partial^{+}_{\beta}(V^{--})]U = -U^{-1}\partial^{+}_{\alpha}\partial^{+}_{\beta}(V^{--})U. \tag{52}$$

Making gauge transformation with the gauge parameter U^{-1}, we get the following representation:

$$A^{+}_{\alpha} = 0, \tag{53}$$

$$V^{++} = -D^{++}(U)U^{-1}, \tag{54}$$

$$V^{--} = -D^{--}(U)U^{-1}, \tag{55}$$

$$A^{-}_{\alpha} = -\partial^{+}_{\alpha}(V^{--}), \tag{56}$$

$$f_{\alpha\beta} = -\partial^{+}_{\alpha}\partial^{+}_{\beta}V^{--}. \tag{57}$$

The identity (45) together with (12) gives us:

$$\begin{aligned}\frac{1}{2}\Delta^2 \log \mathrm{Det}(D^{++} + V^{++}) &= \partial^{+\alpha}\partial^{-}_{\alpha}\partial^{\beta\dot{\beta}}\partial_{\beta\dot{\beta}} \log \mathrm{Det}(D^{++} + V^{++}) \\ &= \frac{1}{2}\int \partial^{+\alpha}\partial^{-}_{\alpha}\partial^{\beta\dot{\beta}}\, \mathrm{tr}\, V^{--}\partial_{\beta\dot{\beta}}(V^{++}).\end{aligned}$$

Since $\partial_{\beta\dot{\beta}} = u^{+}_{\dot{\beta}}\partial^{-}_{\beta} + u^{-}_{\dot{\beta}}\partial^{+}_{\beta}$ and $\partial^{+\beta}(V^{++}) = 0$ we obtain

$$\Delta^2 \log \mathrm{Det}(D^{++} + V^{++}) = \int \partial^{-}_{\alpha}\, \mathrm{tr}\, \partial^{+\alpha}\partial^{+\beta}(V^{--})\partial^{-}_{\beta}(V^{++}).$$

Now we have:

$$\partial^{-}_{\beta}(V^{++}) = [D^{--}, \partial^{+}_{\beta}]V^{++} = -\partial^{+}_{\beta}D^{--}(V^{++}).$$

Using the flatness condition (46), one could replace $D^{--}(V^{++})$ by $\mathcal{D}^{++}(V^{--})$:

$$\mathcal{D}^{++}(V^{--}) = D^{++}(V^{--}) + [V^{++}, V^{--}].$$

Therefore we get:

$$\partial^{-}_{\beta}(V^{++}) = -\partial^{+}_{\beta}\mathcal{D}^{++}(V^{--}) = -\mathcal{D}^{++}\partial^{+}_{\beta}(V^{--}).$$

Taking into account (57), one derives:

$$\Delta^2 \log \mathrm{Det}(D^{++} + V^{++}) = \int \partial^{-\alpha}\, \mathrm{tr}\, f^{\alpha\beta}\mathcal{D}^{++}\partial^{+}_{\beta}(V^{--}).$$

Now from the Bianchi identity $\nabla^{-}_{\alpha}(f^{\alpha\beta}) = \partial^{-}_{\alpha}(f^{\alpha\beta}) + [A^{-}_{\alpha}, f^{\alpha\beta}] = 0$ one can obtain

$$\begin{aligned}
&\Delta^2 \log \mathrm{Det}(D^{++} + V^{++}) \\
&\quad = \int \mathrm{tr}\, f^{\alpha\beta}\nabla^{-}_{\alpha}\mathcal{D}^{++}\partial^{+}_{\beta}(V^{--}) \\
&\quad = \int \mathrm{tr}\, f^{\alpha\beta}[\nabla^{-}_{\alpha}, \mathcal{D}^{++}]\partial^{+}_{\beta}(V^{--}) + \mathrm{tr}\, f^{\alpha\beta}\mathcal{D}^{++}\nabla^{-}_{\alpha}\partial^{+}_{\beta}(V^{--}) \\
&\quad = \int \mathrm{tr}\, f^{\alpha\beta} f_{\alpha\beta} - \mathrm{tr}\,\mathcal{D}^{++}(f^{\alpha\beta}) f_{\alpha\beta} + D^{++}(\mathrm{tr}\, f^{\alpha\beta} f_{\alpha\beta}).
\end{aligned}$$

The second and the third terms are zero due to the relations $\mathcal{D}^{++}(f^{\alpha\beta}) = 0$ and $\int D^{++}(\mathrm{tr}\, f^{\alpha\beta} f_{\alpha\beta}) = 0$. Therefore we get the simple identity

$$\Delta^2 \log \mathrm{Det}(D^{++} + V^{++}) = \int \mathrm{tr}\, f^{\alpha\beta} f_{\alpha\beta}. \tag{58}$$

Taking into account the relation $(F \wedge F) = -\frac{1}{2} f_{\alpha\beta} f^{\alpha\beta}\, \mathrm{vol}_M$ we complete the proof of the theorem.

4. Explicit calculation for one-instanton connection

There is a well known explicit formula for the density of the topological charge of the gauge field describing one-instanton solution of the self-duality equations:

$$2\,\mathrm{tr}\, F \wedge F = -\Delta^2 \log\left(1 + \frac{x^2}{\rho^2}\right)\mathrm{vol}_M. \tag{59}$$

Here the instanton with the center at $x = 0$ has the size ρ. This formula is obviously a particular case of our general formula (43) with:

$$16\pi^2 T = \log\left(1 + \frac{x^2}{\rho^2}\right). \tag{60}$$

In this section we show how our general expression for T reduces to (60). Consider the expansion of the determinant:

$$\log \mathrm{Det}\left(1 + \frac{1}{D^{++}}V^{++}\right) = \sum_{k=1}(-1)^k \frac{1}{k}\int \mathrm{d}^2u\, \mathrm{Tr}\left(\frac{1}{D^{++}}V^{++}\right)^k. \tag{61}$$

Taking into account the simple identity:

$$D^{++}(x_i^+ x^{-j}) = x_i^+ x^{+j}, \tag{62}$$

let us analyze first terms of the expansion:

$$\frac{1}{D^{++}} \frac{x_i^+ x^{+j}}{\rho^2} + \frac{1}{D^{++}} \frac{x_i^+ x^{+j}}{\rho^2} \frac{1}{D^{++}} \frac{x_i^+ x^{+j}}{\rho^2} + \cdots \tag{63}$$

$$= \frac{x_i^+ x^{-j}}{\rho^2} + \frac{1}{D^{++}} \frac{x_i^+ x^{+l} x_l^- x^{+j}}{\rho^4} + \cdots \tag{64}$$

$$= \frac{x_i^+ x^{-j}}{\rho^2} + \frac{x_i^+ x^{-j} |x|^2}{\rho^4} + \cdots . \tag{65}$$

Here we have used the relations:

$$x_j^+ x^{+j} = 0, \tag{66}$$

$$x_j^+ x^{-j} = |x|^2. \tag{67}$$

It is clear that different terms in the expansion are connected by simple relations. Taking the integral over u variable we get for the full series:

$$16\pi^2 T = \sum (-1)^k \frac{1}{k} \frac{|x|^{2k}}{\rho^{2k}} = \log\left(1 + \frac{|x|^2}{\rho^2}\right). \tag{68}$$

Acknowledgements

During the course of this work, the authors benefited from helpful conversations with A. Levin, V. Rubtsov. The work of AG was partially supported by RFBR grant 98-01-00328 and Grant for the Support of Scientific Schools 00-15-96557. The work of AK was partially supported by Svenska Institute through the Nya Visby Program 2507/2001, by RFBR grant 99-01-01254 and Grant for the Support of Scientific Schools 00-15-99296.

Appendix A. Determinants of Cauchy–Riemann operators over Riemann surface

In this section we recall basic facts about the determinants of $\bar{\partial}_A$-operators (chiral determinants) and prove the identities used in the main body of the paper.

Let M be a compact one-dimensional complex manifold and E be a smooth vector bundle over M. Let ∇_A be a holomorphic connection. We denote $(1, 0)$ and $(0, 1)$ components of ∇_A as ∂_A and $\bar{\partial}_A$ respectively and identify the affine space $\mathcal{A}$ of $\bar{\partial}_A$-operators with the space of holomorphic structures in E.

Let Δ_A be a Laplace operator written as follows: $\Delta_A = \bar{\partial}_A^* \bar{\partial}_A$. Assume that $\bar{\partial}_A$ is invertible and ∂_A is conjugated to $\bar{\partial}_A$ with respect to a suitable hermitian metric.

Theorem 2 (Quillen [7]). *Let A_0 be a base point. Then there exists an unique up to a constant holomorphic function* $\mathrm{Det}(A_0, A)$ on $\mathcal{A}$ such that

$$\mathrm{Det}\,\Delta_A = e^{-\|A-A_0\|^2}|\mathrm{Det}(A_0, A)|^2. \tag{A.1}$$

Here $\mathrm{Det}\,\Delta_A = \exp(-\partial/\partial s_{|s=0}\,\mathrm{Tr}\,\Delta_A^{-s})$ is ζ-regularized determinant of Δ_A and

$$\|B\|^2 = \frac{\mathrm{i}}{2\pi}\int_M \mathrm{tr}\,B\bar{B}.$$

Proof.

$$\delta_{\bar{A}}\log\mathrm{Det}\Delta_A = \frac{\partial}{\partial s}_{|s=0} s\,\mathrm{Tr}(\Delta_A^{-s}\Delta_A^{-1}\delta\Delta_A) = \mathrm{Tr}(\Delta_A^{-s}\bar{\partial}_A^{-1}\delta\bar{A})_{|s=0}.$$

Taking the variation in the form $\delta\bar{A} = \bar{\partial}_A(\epsilon) = [\bar{\partial}_A, \epsilon]$ one gets

$$\mathrm{Tr}(\Delta_A^{-s}\bar{\partial}_A^{-1}\delta\bar{A})_{|s=0} = \mathrm{Str}(\Delta_A^{-s}\bar{\partial}_A^{-1}\delta\bar{A})_{|s=0} = \mathrm{Str}(\Delta_A^{-s}\epsilon).$$

Simple calculation shows that the regular value of $\langle x|\Delta_A^{-s}|x\rangle$ at $s = 0$ is equal to $(1/2\pi\mathrm{i})$ $(F_A + \frac{1}{2}F_{\tau_M})(x)$, where τ_M is holomorphic tangent bundle. Thus we have

$$\delta_{\bar{A}}\log\mathrm{Det}\Delta_A = \frac{1}{2\pi i}\int_M \mathrm{tr}\,F_A\epsilon + \frac{1}{2}F_{\tau_M}\,\mathrm{tr}\,\epsilon.$$

Hence

$$\delta_A\delta_{\bar{A}}\log\mathrm{Det}\Delta_A = -\frac{1}{2\pi\mathrm{i}}\int_M \mathrm{tr}\,\bar{\partial}_A(\delta A)\epsilon = -\frac{\mathrm{i}}{2\pi}\int_M \mathrm{tr}\,\delta A\delta\bar{A}.$$

Therefore there is a holomorphic function $\mathrm{Det}(A_0, A)$ on $\mathcal{A}$ such that

$$\mathrm{Det}\,\Delta_A = e^{-\|A-A_0\|^2}|\mathrm{Det}(A_0, A)|^2.$$

Denote $\partial = \partial_{A_0}$, $\bar{\partial} = \bar{\partial}_{A_0}$. Making infinitesimal gauge transformations

$$\delta A = \partial_A(\epsilon) = \partial(\epsilon) + [A, \epsilon], \tag{A.2}$$

$$\delta\bar{A} = \bar{\partial}_A(\epsilon) = \bar{\partial}(\epsilon) + [\bar{A}, \epsilon], \tag{A.3}$$

in the formula (A.1) one gets

$$\delta_\epsilon\log\mathrm{Det}\,\Delta_A = -\frac{\mathrm{i}}{2\pi}\int \mathrm{tr}\,\delta A\bar{A} - \frac{\mathrm{i}}{2\pi}\int \mathrm{tr}\,A\delta\bar{A} + \delta_\epsilon\log\mathrm{Det}(\partial + A)$$
$$+\delta_\epsilon\log\mathrm{Det}(\bar{\partial} + \bar{A}).$$

Since the determinant of Δ_A is gauge invariant we have:

$$\delta_\epsilon\log\mathrm{Det}(\partial + A) + \delta_\epsilon\log\mathrm{Det}(\bar{\partial} + \bar{A}) = \frac{\mathrm{i}}{2\pi}\int \mathrm{tr}\,\partial_A(\epsilon)\bar{A} + \frac{\mathrm{i}}{2\pi}\int \mathrm{tr}\,A\bar{\partial}_A(\epsilon).$$

The simple identity $\mathrm{tr}[A,\epsilon]\bar{A} + \mathrm{tr}\, A[\bar{A},\epsilon] = 0$ leads to

$$\delta_\epsilon \log \mathrm{Det}(\partial + A) + \delta_\epsilon \log \mathrm{Det}(\bar{\partial} + \bar{A}) = \frac{\mathrm{i}}{2\pi}\int \mathrm{tr}\, \partial(\epsilon)\bar{A} + \frac{\mathrm{i}}{2\pi}\int \mathrm{tr}\, A\bar{\partial}(\epsilon).$$

Both left and right hand sides of the formula are decomposed into the sum of the holomorphic and antiholomorphics parts. Considering the antiholomorphic part we obtain the variation formula:

$$\delta_\epsilon \log \mathrm{Det}(\bar{\partial} + \bar{A}) = \frac{\mathrm{i}}{2\pi}\int \mathrm{tr}\, \partial(\epsilon)\bar{A}. \tag{A.4}$$

Now let us define the variation derivative $A(\bar{A})$ of $\log \mathrm{Det}(\bar{\partial} + \bar{A})$ as:

$$\delta_{\bar{A}} \log \mathrm{Det}(\bar{\partial} + \bar{A}) = \frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2 z(\delta\bar{A} \wedge A(\bar{A})). \tag{A.5}$$

Expressing the equation (A.4) in terms of $A(\bar{A})$ and $\bar{A}$ we get the condition:

$$\bar{\partial}A - \partial\bar{A} + [\bar{A}, A] = 0. \qquad \square \tag{A.6}$$

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